Further Properties of Wireless Channel Capacity

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Outline

Instantaneous Capacity

- Wireless fading channels are time variant and wireless channel capacity is a stochastic process [Tse, 2005]
- The instantaneous capacity of the channel at time t can be expressed as a function of the instantaneous SNR γ_t at this time [Costa and Haykin, 2010]

$$C(t) = \log_2(g(\gamma_t))$$

- Statistical properties of first order and second order have been investigated [Rafiq, 2011, Pätzold, 2011]
  - mean, variance, PDF, CDF, LCR, and ADF

Motivation of this Work

- Capacity and QoS requirements in future wireless communication
  - more data (500 EB), higher data rate (1000$\times$, 100$\times$), and less latency ($<$1ms, round-trip) in 5G [Andrews et al., 2014]
- Instantaneous capacity is not sufficient for use in assessing if data transmission over the channel meets its QoS requirements
  - capacity behavior of average sense
    - ergodic capacity
  - temporal behavior of the capacity
    - LCR, ADF

Fundamental Concepts

- Cumulative capacity

$$S(s,t) \equiv \sum_{i=s+1}^{t} C(i)$$

- Maximum cumulative capacity

$$\overline{S}(0,t) \equiv \sup_{1\le j\le k\le t} S(j,k) = \sup_{1\le j\le k\le t} \left(\sum_{i=j}^{k} C(i) \right)$$

  - forward-looking and backward-looking variations

$$\overrightarrow{S}(0,t) \equiv \sup_{1\le k\le t} \overline{S}(0,k), \quad \overleftarrow{S}(0,t) \equiv \sup_{1\le j\le t} \overline{S}(j,t)$$

Fundamental Concepts (Cont'd)

- Minimum cumulative capacity

$$\underline{S}(0,t) \equiv \inf_{1\leq j\leq k\leq t} S(j,k) = \inf_{1\leq j\leq k\leq t} \left(\sum_{i=j}^{k} C(i) \right)$$

  - forward-looking and backward-looking variations

$$\underset{\rightarrow}{\underline{S}}(0,t) \equiv \inf_{1\leq k\leq t} \underline{S}(0,k), \quad \underset{\leftarrow}{\underline{S}}(0,t) \equiv \inf_{1\leq j\leq t} \underline{S}(j,t)$$

- Range of cumulative capacity

$$R(0,t) \equiv \overline{S}(0,t) - \underline{S}(0,t)$$

Exact Expression

The CDF of the cumulative capacity is expressed as

$$F_{S(s,t)}(x) = \int_{S(s,t)=\sum_{i=s+1}^{t} \log_2(1+\gamma|h_i|^2)\leq x} dF_{\mathbf{H}}(h_{s+1}, h_{s+2}, \dots, h_t),$$

where $F_{\mathbf{H}}(h_{s+1}, h_{s+2}, \dots, h_t)$ is the joint distribution of channel gains, e.g. the multivariate generalized Rician distribution [Beaulieu and Hemachandra, 2011]

$$\begin{aligned} F_{\mathbf{H}}(h_1, h_2, \dots, h_N) &= \int_{t=0}^{\infty} \frac{t^{\frac{m-1}{2}}}{S^{m-1}} \exp(-(t+S^2)) I_{m-1}(2S\sqrt{t}) \\ &\quad \prod_{k=1}^{N} \left[1 - Q_m\left(\frac{\sqrt{t}\sqrt{\sigma_k^2\lambda_k^2}}{\Omega_k}, \frac{h_k}{\Omega_k}\right)\right] dt. \end{aligned}$$

Standard Bounds

The CDF of the cumulative capacity satisfies the following inequalities:

$$F^{l}_{S(s,t)}(r) \leq F_{S(s,t)}(r) \leq F^{u}_{S(s,t)}(r),$$

where

$$F^{u}_{S(s,t)}(r) \equiv \inf_{\sum_{i=s+1}^{t} r_i = r} \left[\sum_{i=s+1}^{t} F_{C(i)}(r_i) \right]_1 ,$$

$$F^{l}_{S(s,t)}(r) \equiv \sup_{\sum_{i=s+1}^{t} r_i = r} \left[\sum_{i=s+1}^{t} F_{C(i)}(r_i) - (t-s-1) \right]^{+} .$$

Improved Bounds

Let $F_1 = \ldots = F_n =: F$ be distribution functions on $\mathbb{R}_+$. Then for any $s \geq 0$ it holds that [Puccetti and Rüschendorf, 2012]

$$\begin{aligned} M_n^+(s) &\leq D(s) = \inf_{u<s/n} \min\left\{ \frac{n \int_u^{s-(n-1)u} \overline{F}(t)dt}{s-nu}, 1 \right\}, \\ m_n^+(s) &\geq d(s) = \sup_{u>s/n} \max\left\{ \frac{n \int_u^{s-(n-1)u} \overline{F}(t)dt}{s-nu} - n + 1, 0 \right\}, \end{aligned}$$

where

$$\begin{aligned} M_n^+(t) &= \sup\left\{ P\left(\sum_{i=1}^n X_i \geq t\right); X_i \sim F_i, 1 \leq i \leq n \right\}, \\ m_n^+(t) &= \inf\left\{ P\left(\sum_{i=1}^n X_i > t\right); X_i \sim F_i, 1 \leq i \leq n \right\}. \end{aligned}$$

Comonotonicity

- The set $A \subseteq \mathbb{R}^n$ is said to be comonotonic if for any $\underline{x} \leq \underline{y}$ or $\underline{y} \leq \underline{x}$ holds, where $\underline{x} \leq \underline{y}$ denotes the componentwise order, i.e., $x_i \leq y_i$ for all $i = 1, 2, \ldots, n$. [Dhaene et al., 2002]
- In the special case that all marginal distribution functions are identical $F_{C(i)} \sim F_C$, comonotonicity of $C(i)$ is equivalent to saying that $C(s+1) = C(s+2), \ldots, = C(t)$ holds almost surely [Dhaene et al., 2002], i.e.,

$$F_{S(s,t)}(x) = F_C\left(\frac{x}{t-s}\right).$$

Independence

- If $C(i)$ and $C(j)$, $i \neq j$, are independent, $f_{S(s,t)} = f_{C(s+1)} * \ldots * f_{C(t)}$, where $*$ denotes the convolution operation, namely, $F_{S(s,t)}(x) = \int_{-\infty}^{x} f_{S(s,t)}(y)dy$.
- According to the central limit theorem, $F_{S(s,t)}(x)$ approaches a normal distribution [Papoulis and Pillai, 2002], i.e.,

$$F_{S(s,t)}(x) \approx G\left(\frac{x - E[S(s,t)]}{\sigma^2[S(s,t)]}\right).$$

- For identical marginals $F_{C(i)} \sim F_C$, according to the Markov inequality

$$P\{L_t \geq \mu\} \leq \frac{1}{\mu}\mathbb{E}[L_t] = \frac{1}{\mu}, \; P\{S_t \geq x\} \leq e^{\theta x - t\kappa(\theta)},$$

where $\kappa(\theta) = \log \mathbb{E}e^{\theta C(i)} = \log \int e^{\theta x} F(dx)$, $L_t = e^{\theta S_t - t\kappa(\theta)}$, and L_t is a mean-one martingale [Asmussen, 2003].

Markov Process

- For a Markov additive process, denote matrix $\widehat{\mathbf{F}}[\theta]$ with $ij$the element $\widehat{F}^{(ij)}[\theta] =: \int e^{\theta x} F^{(ij)}(dx)$, where $F_{ij}(dx) = \mathbb{P}_{i,0}(J_1 = j, Y_1 \in dx)$, $Y_n = S_n - S_{n-1}$. By Perron-Frobenius theory, the matrix $\widehat{\mathbf{F}}[\theta]$ has a positive real eigenvalue with maximal absolute value $e^{\kappa(\theta)}$ and the corresponding right eigenvector $\mathbf{h}^{(\theta)} = (h_i^{(\theta)})_{i \in E}$, i.e., $\widehat{\mathbf{F}}[\theta]\mathbf{h}^{(\theta)} = e^{\kappa(\theta)}\mathbf{h}^{(\theta)}$. [Asmussen, 2003]
- Let

$$L_n = \frac{h^{(\theta)}(J_n)}{h^{(\theta)}(J_0)} e^{-\theta S_n + n\kappa(\theta)}, \ \underline{L}_n = \frac{\min_n(h^{(\theta)}(J_n))}{h^{(\theta)}(J_0)} e^{-\theta S_n + n\kappa(\theta)},$$

according to Markov inequality [Gallager, 2013]

$$\begin{aligned} P\{\underline{L}_n \geq \mu\} &\leq \frac{1}{\mu}\mathbb{E}[\underline{L}_n] \leq \frac{1}{\mu}, \\ P\{S_n \geq \alpha\} &\leq e^{-n\kappa(\theta)+\theta\alpha} h^{(\theta)}(J_0)/\min_n(h^{(\theta)}(J_n)). \end{aligned}$$

Non-Granger Causality Assumption

- Non-Granger causality refers to a multivariate dynamic system in which each variable is determined by its own lagged values and no further information is provided by the lagged values of the other variables.
- Then the copula function representing the dependence structure among the running maxima (minima) at time t_n is the same copula function (survival copula function) representing dependence among the levels at the same time [Cherubini and Romagnoli, 2010].

A Lower Bound for Maximum Cumulative Capacity

The CDF of the maximum cumulative capacity is bounded by

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le i\le t} \overline{S}(i) \le x\right) &= \mathbb{P}\left(S(1)\le x, S(2)\le x, \ldots, S(t)\le x\right) \\
&\ge \mathbb{P}\left(\max C(1)\le x, \max_{1\le i\le 2} C(i) \le \frac{x}{2}, \ldots, \max_{1\le i\le t} C(i) \le \frac{x}{t}\right) \\
&= C\left(F_{M_1}(x), F_{M_2}\left(\frac{x}{2}\right), \ldots, F_{M_t}\left(\frac{x}{t}\right)\right) \\
&= C\left(F(x), F\left(\frac{x}{2}, \frac{x}{2}\right), \ldots, F\left(\frac{x}{t}, \frac{x}{t}, \ldots, \frac{x}{t}\right)\right),
\end{aligned}$$

where $F(x_1, x_2, \ldots, x_t) = C(F_{C(1)}(x_1), F_{C(2)}(x_2), \ldots, F_{C(t)}(x_t))$.

An Upper Bound for Minimum Cumulative Capacity

The CDF of the minimum cumulative capacity is bounded by

$$\begin{aligned}
\mathbb{P}\left(\inf_{0\le i\le t} \underline{S}(i) \le x\right) &= 1-\mathbb{P}\left(S(1)>x, S(2)>x,\ldots,S(t)>x\right) \\
&\le 1-\mathbb{P}\left(\min C(1)>x, \min_{1\le i\le 2} C(i)>\frac{x}{2},\ldots,\min_{1\le i\le t} C(i)>\frac{x}{t}\right) \\
&= 1-\overline{C}\left(\overline{F}_{m_1}(x), \overline{F}_{m_2}\left(\frac{x}{2}\right),\ldots,\overline{F}_{m_t}\left(\frac{x}{t}\right)\right) \\
&= 1-\overline{C}\left(\overline{F}(x), \overline{F}\left(\frac{x}{2},\frac{x}{2}\right),\ldots,\overline{F}\left(\frac{x}{t},\frac{x}{t},\ldots,\frac{x}{t}\right)\right),
\end{aligned}$$

where $F(x_1,x_2,\ldots,x_t)=C(F_{C(1)}(x_1),F_{C(2)}(x_2),\ldots,F_{C(t)}(x_t))$.

Independence

- For identical marginals $F_{C(i)} \sim F_C$, the cumulant generating function and the likelihood ratio are expressed as [Asmussen, 2003]

$$\begin{aligned} \kappa(\theta) &= \log \mathbb{E} e^{\theta C(i)} = \log \int e^{\theta x} F(dx), \\ L_t &= e^{\theta S_t - t\kappa(\theta)}, \end{aligned}$$

where L_t is a mean-one martingale.

- Let the Lundberg equation $\kappa(\theta) = 0$ and assume the existence of a solution $\theta > 0$, then [Asmussen, 2003]

$$P\left\{ \sup_{t \geq 0} S_t \geq x \right\} \leq e^{-\theta x},$$

for all $x \geq 0$.

Markov Process

Let $\tau(u) = \inf\{t > 0 : S_t > u\}$, $I(u) = J_{\tau(u)}$, $\xi(u) = S_{\tau(u)} - u$, $M = \sup_{t \geq 0} S_t$. Let the Lundberg equation $\kappa(\theta) = 0$ and assume the existence of a solution $\theta > 0$. Then [Asmussen, 2003, Asmussen and Albrecher, 2010]

$$\begin{aligned} \mathbb{P}_i(M > u) &= \mathbb{P}_i(\tau(u) < \infty) = \mathbb{E}_{i,\theta}\left[\frac{h_{J_0}^{(\theta)}}{h_{J_{\theta(u)}}^{(\theta)}} e^{-\theta S_{\tau(u)}}; \tau(u) < \infty \right] \\ &= e^{-\theta u} \mathbb{E}_{i,\theta}\left[\frac{h_i^{(\theta)}}{h_{I(u)}^{(\theta)}} e^{-\theta \xi(u)} \right], \\ \mathbb{P}(M > u) &= \sum_i \pi_i \mathbb{P}_i. \end{aligned}$$

Markov Process (Cont'd)

- According to Lundberg's inequility [Asmussen and Albrecher, 2010]

$$\mathbb{P}_i(M > u) \quad \leq \quad \frac{h_i^{(\theta)}}{\min_{j \in E} h_j^{(\theta)}} e^{-\theta u}.$$

- The above inequality can be improved together with a lower bound. Let

$$C_- \quad = \quad \min_{j \in E} \frac{1}{h_j^{(\theta)}} \cdot \inf_{x \geq 0} \frac{\overline{B}_j(x)}{\int_x^\infty e^{\theta(y-x)} B_j(dy)},$$

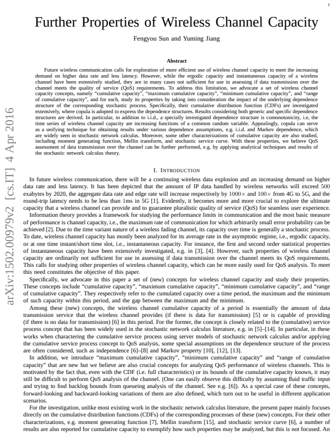
$$C_+ \quad = \quad \max_{j \in E} \frac{1}{h_j^{(\theta)}} \cdot \sup_{x \geq 0} \frac{\overline{B}_j(x)}{\int_x^\infty e^{\theta(y-x)} B_j(dy)},$$

where B_j is the distribution of the increment. Then for all $j \in E$ and all $u \geq 0$,

$$C_- h_i^{(\theta)} e^{-\theta u} \leq \mathbb{P}_i(M > u) \leq C_+ h_i^{(\theta)} e^{-\theta u}.$$

Conclusion

- Advocation of a set of wireless channel capacity concepts
- Analysis of the advocated concepts with focus on CDF
- Copula as a unifying technique of analysis considering dependence (see the paper on arXiv)
- Other characterizations, e.g, MGF, MT, SSC (see the paper on arXiv)
- On-going work
  - analysis of backward-looking variations
  - range as a measure of tightness of cumulative capacity bounds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Further Properties of Wireless Channel Capacity

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Abstract

Future wireless communication calls for exploration of more efficient use of wireless channel capacity to meet the increasing demand on higher data rate and less latency. However, while the ergodic capacity and instantaneous capacity of a wireless channel have been extensively studied, they are in many cases not sufficient for use in assessing if data transmission over the channel meets the quality of service (QoS) requirements. To address this limitation, we advocate a set of wireless channel capacity concepts, namely “cumulative capacity”, “maximum cumulative capacity”, “minimum cumulative capacity”, and “range of cumulative capacity”, and for each, study its properties by taking into consideration the impact of the underlying dependence structure of the corresponding stochastic process. Specifically, their cumulative distribution function (CDFs) are investigated extensively, where copula is adopted to express the dependence structures. Results considering both generic and specific dependence structures are derived. In particular, in addition to i.i.d., a specially investigated dependence structure is comonotonicity, i.e, the time series of wireless channel capacity are increasing functions of a common random variable. Appealingly, copula can serve as a unifying technique for obtaining results under various dependence assumptions, e.g. i.i.d. and Markov dependence, which are widely seen in stochastic network calculus. Moreover, some other characterizations of cumulative capacity are also studied, including moment generating function, Mellin transform, and stochastic service curve. With these properties, we believe QoS assessment of data transmission over the channel can be further performed, e.g. by applying analytical techniques and results of the stochastic network calculus theory.

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I. Introduction

In future wireless communication, there will be a continuing wireless data explosion and an increasing demand on higher data rate and less latency. It has been depicted that the amount of IP data handled by wireless networks will exceed 500 exabytes by 2020, the aggregate data rate and edge rate will increase respectively by 1000× and 100× from 4G to 5G, and the round-trip latency needs to be less than 1ms in 5G [1]. Evidently, it becomes more and more crucial to explore the ultimate capacity that a wireless channel can provide and to guarantee pluralistic quality of service (QoS) for seamless user experience.

Information theory provides a framework for studying the performance limits in communication and the most basic measure of performance is channel capacity, i.e., the maximum rate of communication for which arbitrarily small error probability can be achieved [2]. Due to the time variant nature of a wireless fading channel, its capacity over time is generally a stochastic process. To date, wireless channel capacity has mostly been analyzed for its average rate in the asymptotic regime, i.e., ergodic capacity, or at one time instant/short time slot, i.e., instantaneous capacity. For instance, the first and second order statistical properties of instantaneous capacity have been extensively investigated, e.g. in [3], [4]. However, such properties of wireless channel capacity are ordinarily not sufficient for use in assessing if data transmission over the channel meets its QoS requirements. This calls for studying other properties of wireless channel capacity, which can be more easily used for QoS analysis. To meet this need constitutes the objective of this paper.

Specifically, we advocate in this paper a set of (new) concepts for wireless channel capacity and study their properties. These concepts include “cumulative capacity”, “maximum cumulative capacity”, “minimum cumulative capacity”, and “range of cumulative capacity”. They respectively refer to the cumulated capacity over a time period, the maximum and the minimum of such capacity within this period, and the gap between the maximum and the minimum.

Among these (new) concepts, the wireless channel cumulative capacity of a period is essentially the amount of data transmission service that the wireless channel provides (if there is data for transmission) [5] or is capable of providing (if there is no data for transmission) [6] in this period. For the former, the concept is closely related to the (cumulative) service process concept that has been widely used in the stochastic network calculus literature, e.g. in [5]–[14]. In particular, in these works when charactering the cumulative service process using server models of stochastic network calculus and/or applying the cumulative service process concept to QoS analysis, some special assumptions on the dependence structure of the process are often considered, such as independence [6]–[8] and Markov property [10], [12], [13].

In addition, we introduce “maximum cumulative capacity”, “minimum cumulative capacity” and “range of cumulative capacity” that are new but we believe are also crucial concepts for analyzing QoS performance of wireless channels. This is motivated by the fact that, even with the CDF (i.e. full characteristics) or its bounds of the cumulative capacity known, it may still be difficult to perform QoS analysis of the channel. (One can easily observe this difficulty by assuming fluid traffic input and trying to find backlog bounds from queueing analysis of the channel. See e.g. [6]). As a special case of these concepts, forward-looking and backward-looking variations of them are also defined, which turn out to be useful in different application scenarios.

For the investigation, unlike most existing work in the stochastic network calculus literature, the present paper mainly focuses directly on the cumulative distribution functions (CDFs) of the corresponding processes of these (new) concepts. For their other characterizations, e.g. moment generating function [7], Mellin transform [15], and stochastic service curve [6], a number of results are also reported for cumulative capacity to exemplify how such properties may be analyzed, but this is not focused. An

Thank you for your attention!
Questions?

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